Published by Institute of Physics Publishing for SISSA

Received: *July 4, 2007*
Revised: *September 7, 2007*
Accepted: *September 25, 2007*
Published: *October 5, 2007*

Counterterms in dimensionally continued AdS gravity

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Abstract: We revise two regularization mechanisms for Lovelock gravity with AdS asymptotics. The first one corresponds to the Dirichlet counterterm method, where local functionals of the boundary metric are added to the bulk action on top of a Gibbons-Hawking-Myers term that defines the Dirichlet problem in gravity. The generalized Gibbons-Hawking term can be found in any Lovelock theory following the Myers' procedure to achieve a well-posed action principle for a Dirichlet boundary condition on the metric, which is proved to be equivalent to the Hamiltonian formulation for a radial foliation of spacetime. In turn, a closed expression for the Dirichlet counterterms does not exist for a generic Lovelock gravity. The second method supplements the bulk action with boundary terms which depend on the extrinsic curvature (Kounterterms), and whose explicit form is independent of the particular theory considered.

In this paper, we use Dimensionally Continued AdS Gravity (Chern-Simons-AdS in odd and Born-Infeld-AdS in even dimensions) as a toy model to perform the first explicit comparison between both regularization prescriptions. This can be done thanks to the fact that, in this theory, the Dirichlet counterterms can be readily integrated out from the divergent part of the Dirichlet variation of the action.

The agreement between both procedures at the level of the boundary terms suggests the existence of a general property of any Lovelock-AdS gravity: intrinsic counterterms are generated as the difference between the Kounterterm series and the corresponding Gibbons-Hawking-Myers term.

Keywords: AdS-CFT Correspondence, Chern-Simons Theories.

JHEP10(2007)028

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Contents

1. Introduction **1**

2. Dirichlet problem in Lovelock gravity **4**

3. Dirichlet counterterms **9**
3.1 Chern-Simons-AdS 9
3.2 Born-Infeld-AdS 12

4. Kounterterms **13**
4.1 Chern-Simons-AdS 13
4.2 Born-Infeld-AdS 17

5. Conclusions **21**

A. Useful identities **22**

1. Introduction

Lovelock gravity [1] has recently attracted great interest in theoretical physics as higher-curvature terms have been shown to appear in the low-energy limit of String Theory as corrections to Einstein-Hilbert action.

Lovelock gravity in $D = d + 1$ dimensions is described by the action

$$I = \kappa \sum_{p=0}^{[(D-1)/2]} \alpha_p \, I^{(p)} \,, \tag{1.1}$$

where $I^{(p)}$ corresponds to the dimensional continuations of p-dimensional Euler density, i.e.,

$$I^{(p)} = \int\limits_{M_D} \varepsilon_{A_1 \cdots A_D} \, \hat{R}^{A_1 A_2} \cdots \hat{R}^{A_{2p-1} A_{2p}} e^{A_{2p+1}} \cdots e^{A_D} \,, \tag{1.2}$$

that carries an arbitrary weight factor α_p and κ is a gravitational constant. The vielbein $e^A = e^A_\mu \, dx^\mu$ is related to the spacetime metric by $G_{\mu\nu} = \eta_{AB} \, e^A_\mu \, e^B_\nu$, and $\hat{R}^{AB} = d\omega^{AB} + \omega^{AC} \omega_C{}^B$ is the Lorentz curvature associated to the spin connection 1-form $\omega^{AB} = \omega^{AB}_\mu \, dx^\mu$. The curvature 2-form can be expressed in terms of the spacetime Riemann tensor as $\hat{R}^{AB} = \frac{1}{2} \hat{R}^{\alpha\beta}_{\mu\nu} \, e^A_\alpha \, e^B_\beta \, dx^\mu dx^\nu$. The sets $\{A, B, \ldots\}$ and $\{\mu, \nu, \ldots\}$ label tangent space and spacetime indices, respectively. The tensorial equivalence of the action $I^{(p)}$ reads

$$I^{(p)} = -\frac{(D-2p)!}{2^p} \int\limits_{M_D} d^D x \, \sqrt{-G} \, \delta^{[\nu_1 \cdots \nu_{2p}]}_{[\mu_1 \cdots \mu_{2p}]} \, \hat{R}^{\mu_1 \mu_2}_{\nu_1 \nu_2} \cdots \hat{R}^{\mu_{2p-1} \mu_{2p}}_{\nu_{2p-1} \nu_{2p}} \,, \tag{1.3}$$

JHEP10(2007)028

where the totally-antisymmetric Kronecker delta and its properties are given in appendix A. Because the action I is a linear combination of all dimensionally continued lower-dimensional Euler densities, the derived equations of motion are at most of second order in the metric, what frees this theory from ghosts when expanded around a flat background [2]. General covariance, together with second-order field equations, are the basic features of General Relativity generalized by Lovelock gravity to higher dimensions. The theory also possesses exact solutions describing black holes [3], whose thermodynamic behavior resembles the one of Einstein-Hilbert black holes with a modified entropy that is no longer proportional to the horizon's area [4].

Further physical input is in general required to select sensible theories among Lovelock gravities (1.1). For instance, a series of inequivalent gravity actions has been presented in [5], demanding the existence of a unique anti-de Sitter (AdS) vacuum. In particular, Chern-Simons-AdS gravity in odd dimensions [6] and Born-Infeld-AdS gravity in even dimensions — often collectively referred to as *Dimensionally Continued Gravity* [7] —, feature a symmetry enhancement from local Lorentz to AdS group, that leaves the gravitational constant κ and the AdS radius ℓ as the only free parameters in the theory.

As in standard gravity, Lovelock action with cosmological constant is divergent in the infrared region and needs to be regularized. In the AdS/CFT approach [8] to the regularization problem, the finiteness of Einstein-Hilbert action is achieved by the procedure known as holographic renormalization [9–12]. For a fixed boundary data $g_{(0)ij}$, this algorithm reconstructs the spacetime metric solving iteratively the field equations in the Fefferman-Graham frame [13]

$$ds^2 = G_{\mu\nu}\, dx^\mu dx^\nu = \frac{\ell^2}{4\rho^2}\, d\rho^2 + \frac{1}{\rho}\, g_{ij}(x,\rho)\, dx^i dx^j \,. \tag{1.4}$$

Here, $g_{ij}(x,\rho)$ is regular at the conformal boundary $\rho = 0$, so that it can be expanded in its vicinity as

$$g_{ij}(x,\rho) = g_{(0)ij}(x) + \rho\, g_{(1)ij}(x) + \rho^2 g_{(2)ij}(x) + \cdots \,. \tag{1.5}$$

This method results in the addition of boundary terms $\mathcal{L}_{\rm ct}$ to the bulk action (supplemented by the Gibbons-Hawking term [14]), that are local functionals of the boundary metric $h_{ij} = g_{ij}/\rho$, the intrinsic curvature $R^{ij}_{kl}(h)$ and its covariant derivative $\nabla_m R^{ij}_{kl}$. This construction is known as Dirichlet counterterms procedure, what achieves a regularized action [15, 16]

$$I_{\rm reg} = -\frac{1}{16\pi G}\int\limits_M d^{d+1}x\sqrt{-G}\left(\hat{R} - 2\Lambda\right) - \frac{1}{8\pi G}\int\limits_{\partial M} d^dx\,\sqrt{-h}K + \int\limits_{\partial M} d^dx\mathcal{L}_{\rm ct}(h, R(h), \nabla R(h)). \tag{1.6}$$

In the above formula, K is the trace of the extrinsic curvature.

However, the intrinsic regularization defined by this method becomes technically involved in higher dimensions because of the forbidding complexity of the equations for the coefficients $g_{(k)}$ $(1 \leq k \leq \left[\frac{d}{2}\right])$ and the plethora of possible covariant counterterms one could construct on the boundary.

JHEP10(2007)028

For higher-curvature theories, holographic renormalization procedure would be even more cumbersome due to the highly non-linear behavior of the equations of motion. In fact, the regularization of quadratic curvature gravities has been carried out only in particular cases by adding covariant local counterterms that are not necessarily dictated by the holographic renormalization procedure [17]. For Einstein-Gauss-Bonnet AdS gravity (the particular quadratic combination of the curvature given by $p = 2$ in eq. (1.1)), this approach provides the answer only for the five-dimensional case [18]. Thus, it still leaves the open question on the form of the counterterms in higher-dimensional Einstein-Gauss-Bonnet AdS, let alone in a generic Lovelock gravity. Furthermore, in Dimensionally Continued Gravity, the AdS vacuum is a zero of maximal degree in the field equations, such that the first non-trivial relation for the coefficients $g_{(k)}$ in (1.5) will just appear at much higher order in ρ than the linear one.

Whichever the explicit form of the counterterms $\mathcal{L}_{\rm ct}$ may be for Lovelock-AdS gravity, the action (1.6) has to be promoted to the form

$$I_{\rm reg} = I + \kappa \int\limits_{\partial M} d^d x\, \beta_d + \int\limits_{\partial M} d^d x\, \mathcal{L}_{\rm ct}(h, R(h), \nabla R(h))\,, \tag{1.7}$$

such that the generalized Gibbons-Hawking term β_d defines a variational principle for a Dirichlet boundary condition on the metric for the action I in eq. (1.1), what is left unchanged by the addition of intrinsic counterterms. As we will shown in detail below, the on-shell variation of the first two terms in eq. (1.7) adopts the canonical form $\delta I = \int_{\partial M} d^d x\, \pi^{ij} \delta h_{ij}$, where π^{ij} corresponds to the momenta in a radial Hamiltonian formulation for Lovelock gravity. Therefore, the role of the counterterms $\mathcal{L}_{\rm ct}$ is cancelling the divergences in the canonical momenta, but it also means that the series should be obtained from the integration of the divergent part of the Hamiltonian variation in any gravity theory. This has been proved in ref. [19], and allowed to recover the counterterm series in the Einstein-Hilbert case from the action of the dilatations on the gravity fields expansion. Such strategy might also be applied to higher curvature theories but, in practice, such procedure for Lovelock gravity could be much more complicated.

In view of the above arguments, it is quite remarkable that a universal regularization prescription for any Lovelock theory with AdS asymptotics can be provided using boundary terms with explicit dependence on the extrinsic curvature K_{ij}, also known as Kounterterms series [20]

$$\mathcal{I}_{\rm reg} = I + c_d \int\limits_{\partial M} d^d x\, B_d(h, R(h), K)\,. \tag{1.8}$$

Due to a profound connection to topological invariants (Euler term) and Chern-Simons forms, the explicit form of this series only distinguishes even from odd dimensions. The construction of the boundary terms B_d does not make use of the expansion in the metric (1.5). Therefore, for a given dimension, the Kounterterms expression remains the same regardless the particular Lovelock gravity considered, even for Einstein-Hilbert [21, 22] and Einstein-Gauss-Bonnet theories [23]. Only the value of the coupling constant c_d is consistently tuned to achieve a well-posed action principle in a given Lovelock-AdS theory.

JHEP10(2007)028

The agreement between the proposal defined by eq. (1.8) with the standard regularization method, has been found — when the latter exists at all — at the level of the conserved quantities and Euclidean action for asymptotically AdS (AAdS) solutions. In Einstein-Hilbert gravity, a direct comparison between both procedures has been worked out in $2+1$ dimensions, showing that the corresponding boundary prescriptions differs at most by a topological invariant [24]. For higher dimensions, attempting a similar strategy would be in general very involved and not particularly enlightening.

On the other hand, one might expect that further insight on this problem would come out from other Lovelock theories, especially in view of the fact that the form of B_d is universal. But, unfortunately, in many cases there is no even a counterterms series $\mathcal{L}_{\mathrm{ct}}$ to compare with.

In this paper, we use Dimensionally Continued Gravity as a toy model to perform the first explicit comparison between the intrinsic and extrinsic regularization schemes in all dimensions. This is only due to the fact that, in this theory, the obtention of the Dirichlet counterterms from direct integration of the divergent terms in the variation of the action is remarkably simpler than in any other gravity theory.

This article is organized as follows. In the next section, we consider the Dirichlet problem for an arbitrary Lovelock gravity, where the addition of a generalized Gibbons-Hawking term defines a well-posed variational principle for a Dirichlet boundary condition on the metric. This procedure is shown to reproduce the Hamiltonian variation of the action for a radial foliation of the spacetime. In section 3, for Dimensionally Continued Gravity, the series $\mathcal{L}_{\mathrm{ct}}$ is obtained as a total variation of local terms in the Dirichlet problem of the action. In section 4, we briefly review the Kounterterms construction for Lovelock-AdS, specialized for Dimensionally Continued Gravity. Finally, we show that the Dirichlet counterterms are generated simply taking the difference between the Kounterterms series $c_d B_d$ and the generalized Gibbons-Hawking term $\kappa\beta_d$.

2. Dirichlet problem in Lovelock gravity

In general, a well-defined action principle for gravity considers supplementing the bulk Lagrangian by appropriate boundary terms such that the on-shell action is stationary. This means that the surface terms coming from an arbitrary variation of the action must be cancelled by choosing suitable boundary conditions.

The Dirichlet problem for gravity consists in setting a well-posed action principle by imposing a Dirichlet boundary condition on the metric. For Einstein-Hilbert case, this is achieved by adding the Gibbons-Hawking boundary term [14] to the bulk action. The systematic construction of boundary terms that defines the Dirichlet problem in Lovelock gravity was carried out by Myers in ref. [25].

Let us briefly review this formalism. The Einstein-Hilbert term (that corresponds to $p=1$ in (1.3)),

$$I^{(1)} = \int\limits_{M_D} \varepsilon_{A_1\cdots A_D}\, \hat{R}^{A_1A_2} e^{A_3}\cdots e^{A_D}\,, \tag{2.1}$$

JHEP10(2007)028

can be written as the dimensional continuation of the 2-dimensional Euler term $\mathcal{E}_2 = \varepsilon_{AB}\,\hat{R}^{AB}$, which is a topological invariant. The variation of $I^{(1)}$ contributes to the equations of motion and produces a surface term

$$\delta I^{(1)} = \int\limits_{\partial M_D} \varepsilon_{A_1\cdots A_D}\,\delta\omega^{A_1A_2}\,e^{A_3}\cdots e^{A_D}\,. \tag{2.2}$$

In the vicinity of the boundary, we take Gaussian (normal) coordinates

$$ds^2 = G_{\mu\nu}\,dx^\mu dx^\nu = N^2(\rho)\,d\rho^2 + h_{ij}(\rho,x)\,dx^i dx^j\,, \tag{2.3}$$

and the corresponding local orthonormal frame

$$e^1 = Nd\rho\,, \qquad e^a = e_i^a\,dx^i\,, \tag{2.4}$$

with a splitting of the indices $A = (1,a)$ for the tangent space and $\mu = (\rho, i)$ for the spacetime. When torsion vanishes, the spin connection on ∂M_D is

$$\omega^{1a} = K^a = K_i^j\,e_j^a\,dx^i, \qquad \omega^{ab} = \omega_i^{ab}(e_j^c)\,dx^i\,, \tag{2.5}$$

where K_{ij} is the extrinsic curvature, that in the frame (2.3) becomes

$$K_{ij} = -\frac{1}{2N}\,\partial_\rho h_{ij}\,. \tag{2.6}$$

In this coordinate system, the variation (2.2) adopts the form

$$\delta I^{(1)} = -2\int\limits_{\partial M_D} \varepsilon_{a_1\cdots a_d}\,\delta K^{a_1}\,e^{a_2}\cdots e^{a_d}\,, \tag{2.7}$$

where the Levi-Civita tensor at the boundary is defined by $\varepsilon_{1a_1\cdots a_d} = -\varepsilon_{a_1\cdots a_d}$. The above surface term contains the variation of the extrinsic curvature that must be eliminated in the Dirichlet problem.

On the other hand, the integration of $\mathcal{E}_2$ over a two-dimensional manifold without boundary is proportional to the Euler characteristic $\chi(M_2)$. When a boundary is introduced, the Euler theorem assigns a boundary correction given by

$$\int\limits_{M_2} \varepsilon_{AB}\,\hat{R}^{AB} = -4\pi\,\chi(M_2) + \int\limits_{\partial M_2} \varepsilon_{AB}\,\theta^{AB}\,. \tag{2.8}$$

Here $\theta^{AB} = \omega^{AB} - \bar{\omega}^{AB}$ stands for the Second Fundamental Form, i.e., the difference between the dynamic field and a reference spin connection that recovers Lorentz covariance at the boundary. It is common to take $\bar{\omega}^{AB}$ as the spin connection from a product metric that matches the geometry at the boundary, such that

$$\theta^{1a} = K^a\,, \qquad \theta^{ab} = 0\,, \tag{2.9}$$

i.e., only normal components of the Second Fundamental Form are non-vanishing at the boundary [26 – 28]. From the dynamical point of view, variations of both sides of eq. (2.8) produce $\varepsilon_{AB}\,\delta\omega^{AB}$ at the boundary.

JHEP10(2007)028

Thus, in order to cancel the term (2.2) (or equivalently (2.7)), we dimensionally continue the boundary term in eq. (2.8), and obtain the Gibbons-Hawking term

$$\begin{aligned} d^d x\, \beta^{(1)} &= -\varepsilon_{A_1\cdots A_D}\, \theta^{A_1A_2} e^{A_3}\cdots e^{A_D} \\ &= -2\,(D-2)!\, d^d x\, \sqrt{-h}\, K\,. \end{aligned} \tag{2.10}$$

Indeed, the variation of $I^{(1)}_{\rm Dir} = I^{(1)} + \int_{\partial M_D} d^d x \beta^{(1)}$,

$$\delta I^{(1)}_{\rm Dir} = 2\,(D-2) \int\limits_{M_D} \varepsilon_{a_1\cdots a_d}\, \delta e^{a_1} K^{a_2} e^{a_3}\cdots e^{a_d} \tag{2.11}$$

$$= (D-2)! \int\limits_{\partial M_D} d^d x\, \sqrt{-h}\, (h^{-1}\delta h)^j_i\, \left(K^i_j - \delta^i_j\, K\right)\,, \tag{2.12}$$

has a suitable form to impose the Dirichlet boundary condition on the metric h_{ij}.

In dimensions $D \geq 5$, the Gauss-Bonnet term (the second order term in the Lovelock series)

$$\begin{aligned} I^{(2)} &= \int\limits_{M_D} \varepsilon_{A_1\cdots A_D}\, \hat{R}^{A_1A_2} \hat{R}^{A_3A_4} e^{A_5}\cdots e^{A_D} \\ &= -\,(D-4)! \int\limits_{M_D} d^D x\, \sqrt{-G}\, \left(\hat{R}_{\mu\nu\alpha\beta}\, \hat{R}^{\mu\nu\alpha\beta} - 4\hat{R}_{\mu\nu}\, \hat{R}^{\mu\nu} + \hat{R}^2\right)\,, \end{aligned} \tag{2.13}$$

contributes to the bulk dynamics. In order to set the Dirichlet problem for this term, one has to consider the Euler theorem in four dimensions,

$$\int\limits_{M_4} \varepsilon_{ABCD}\, \hat{R}^{AB} \hat{R}^{CD} = 2\,(4\pi)^2\, \chi(M_4) + 2 \int\limits_{\partial M_4} \varepsilon_{ABCD}\, \theta^{AB} \left(R^{CD} + \frac{1}{3}\,(\theta^2)^{CD}\right), \tag{2.14}$$

where $R^{ab} = \frac{1}{2}\, R^{ij}_{kl}(h)\, e^a_i e^b_j\, dx^k dx^l$ is the intrinsic curvature and $R^{1a} = 0$. The dimensional continuation of the second Chern form (i.e., the boundary correction to the Euler characteristic in (2.14)) is [25, 29]

$$\begin{aligned} d^d x\, \beta^{(2)} &= -2\varepsilon_{A_1\cdots A_D}\, \theta^{A_1A_2} \left(R^{A_3A_4} + \frac{1}{3}\,(\theta^2)^{A_3A_4}\right) e^{A_5}\cdots e^{A_D} \\ &= 4\varepsilon_{a_1\cdots a_d}\, K^{a_1} \left(R^{a_2a_3} - \frac{1}{3}\, K^{a_2} K^{a_3}\right) e^{a_4}\cdots e^{a_d} \\ &= -4\,(D-4)!\, d^d x\, \sqrt{-h}\, \delta^{[j_1j_2j_3]}_{[i_1i_2i_3]}\, K^{i_1}_{j_1} \left(\frac{1}{2}\, R^{i_2i_3}_{j_2j_3}(h) - \frac{1}{3}\, K^{i_2}_{j_2} K^{i_3}_{j_3}\right)\,, \end{aligned} \tag{2.15}$$

such that the corresponding Dirichlet variation is

$$\delta I^{(2)}_{\rm Dir} = -2\,(D-4)! \int\limits_{\partial M_D} d^d x\, \sqrt{-h}\, \delta^{[j\, j_1j_2j_3]}_{[i\, i_1i_2i_3]}\, (h^{-1}\delta h)^i_j\, K^{i_1}_{j_1} \left(\frac{1}{2}\, R^{i_2i_3}_{j_2j_3}(h) - \frac{1}{3}\, K^{i_2}_{j_2} K^{i_3}_{j_3}\right)\,. \tag{2.16}$$

JHEP10(2007)028

We have used the Gauss-Codazzi relations at the boundary

$$\hat{R}^{ab} = R^{ab} - K^a K^b \,, \tag{2.17}$$

$$\hat{R}^{1a} = DK^a \,, \tag{2.18}$$

or equivalently

$$\hat{R}^{ij}_{kl} = R^{ij}_{kl}(h) - K^i_k K^j_l + K^i_l K^j_k \,, \tag{2.19}$$

$$\hat{R}^{i\rho}_{jk} = \frac{1}{N}\left(\nabla_j K^i_k - \nabla_k K^i_j\right) \,, \tag{2.20}$$

where $D_i = D_i(\omega)$ and $\nabla_i = \nabla_i(\Gamma)$ are covariant derivatives with respect to the spin connection and Christoffel symbol, respectively.

For arbitrary p, the generalized Gibbons-Hawking term is

$$\begin{aligned} d^d x\, \beta^{(p)} &= -p \int_0^1 dt\, \varepsilon_{A_1 \cdots A_D}\, \theta^{A_1 A_2} \left(R^{A_3 A_4} + t^2 (\theta^2)^{A_3 A_4}\right) \times \cdots \\ &\qquad \cdots \times \left(R^{A_{2p-1} A_{2p}} + t^2 (\theta^2)^{A_{2p-1} A_{2p}}\right) e^{A_{2p+1}} \cdots e^{A_D} \qquad (2.21) \\ &= 2p \int_0^1 dt\, \varepsilon_{a_1 \cdots a_d}\, K^{a_1} \left(R^{a_2 a_3} - t^2 K^{a_2} K^{a_3}\right) \times \cdots \\ &\qquad \cdots \times \left(R^{a_{2p-2} a_{2p-1}} - t^2 K^{a_{2p-2}} K^{a_{2p-1}}\right) e^{a_{2p}} \cdots e^{a_d} \,, \qquad (2.22) \end{aligned}$$

or in tensorial notation

$$\begin{aligned} d^d x\, \beta^{(p)} &= -2p\,(D-2p)!\, d^d x \int_0^1 dt\, \delta^{[j_1 \cdots j_{2p-1}]}_{[i_1 \cdots i_{2p-1}]}\, K^{i_1}_{j_1} \left(\frac{1}{2} R^{i_2 i_3}_{j_2 j_3}(h) - t^2 K^{i_2}_{j_2} K^{i_3}_{j_3}\right) \times \cdots \\ &\qquad \cdots \times \left(\frac{1}{2} R^{i_{2p-2} i_{2p-1}}_{j_{2p-2} j_{2p-1}}(h) - t^2 K^{i_{2p-2}}_{j_{2p-2}} K^{i_{2p-1}}_{j_{2p-1}}\right) . \qquad (2.23) \end{aligned}$$

It is worthwhile noticing that the procedure of dimensional continuation of a given Chern form to define the Dirichlet problem in Lovelock gravity does not work in spacetimes with torsion (Riemann-Cartan theory).

The Dirichlet variation for the p-th term of Lovelock series takes the form

$$\begin{aligned} \delta I^{(p)}_{\text{Dir}} &= -p\,(D-2p)! \int_{\partial M_D} d^d x \sqrt{-h} \int_0^1 dt\, \delta^{[j j_1 \cdots j_{2p-1}]}_{[i i_1 \cdots i_{2p-1}]} \left(h^{-1}\delta h\right)^i_j K^{i_1}_{j_1} \times \qquad (2.24) \\ &\times \left(\frac{1}{2} R^{i_2 i_3}_{j_2 j_3}(h) - t^2 K^{i_2}_{j_2} K^{i_3}_{j_3}\right) \cdots \left(\frac{1}{2} R^{i_{2p-2} i_{2p-1}}_{j_{2p-2} j_{2p-1}}(h) - t^2 K^{i_{2p-2}}_{j_{2p-2}} K^{i_{2p-1}}_{j_{2p-1}}\right) . \end{aligned}$$

As a consequence, the Lovelock action set for the Dirichlet problem is

$$I_{\text{Dir}} = I + \kappa \int_{\partial M_D} d^d x \beta_d \,, \tag{2.25}$$

JHEP10(2007)028

where the boundary term is given by

$$\beta_d = \sum_{p=0}^{[(D-1)/2]} \alpha_p \, \beta^{(p)} \,. \tag{2.26}$$

Finally, the variation of the Dirichlet action can be written as

$$\begin{aligned} \delta I_{\mathrm{Dir}} = & -\kappa \sum_{p=0}^{[(D-1)/2]} \alpha_p \, p \, (D-2p)! \int\limits_{\partial M_D} d^d x \sqrt{-h} \int\limits_0^1 dt \, \delta_{[ii_1 \cdots i_{2p-1}]}^{[jj_1 \cdots j_{2p-1}]} \, (h^{-1}\delta h)^i_j \, K^{i_1}_{j_1} \, \times \\ & \times \left(\frac{1}{2} \, R^{i_2 i_3}_{j_2 j_3}(h) - t^2 K^{i_2}_{j_2} K^{i_3}_{j_3} \right) \cdots \left(\frac{1}{2} \, R^{i_{2p-2} i_{2p-1}}_{j_{2p-2} j_{2p-1}}(h) - t^2 \, K^{i_{2p-2}}_{j_{2p-2}} K^{i_{2p-1}}_{j_{2p-1}} \right) . \end{aligned} \tag{2.27}$$

The parametric integration can be performed explicitly, and using the relation between spacetime and induced Riemann tensors (2.19) produces

$$\begin{aligned} & \int\limits_0^1 dt \, \delta_{[ii_1 \cdots i_{2p-1}]}^{[jj_1 \cdots j_{2p-1}]} \, K^{i_1}_{j_1} \left(\frac{1}{2} \, R^{i_2 i_3}_{j_2 j_3}(h) - t^2 K^{i_2}_{j_2} K^{i_3}_{j_3} \right) \cdots \left(\frac{1}{2} \, R^{i_{2p-2} i_{2p-1}}_{j_{2p-2} j_{2p-1}}(h) - t^2 \, K^{i_{2p-2}}_{j_{2p-2}} K^{i_{2p-1}}_{j_{2p-1}} \right) \\ & \qquad = \frac{1}{2^{p+1}} \, \delta_{[ii_1 \cdots i_{2p-1}]}^{[jj_1 \cdots j_{2p-1}]} \sum_{s=0}^{p-1} \frac{4^{p-s} \, (p-1)!}{s! \, (2p-2s-1)!!} \, \hat{R}^{i_1 i_2}_{j_1 j_2} \cdots \hat{R}^{i_{2s-1} i_{2s}}_{j_{2s-1} j_{2s}} \, K^{i_{2s+1}}_{j_{2s+1}} \cdots K^{i_{2p-1}}_{i_{2p-1}} \,. \end{aligned}$$

It is clear from the last line that the Dirichlet variation agrees with the variation of the action in the Hamiltonian formulation of Lovelock gravity [30] for the radial foliation of spacetime considered in [31],

$$\delta I_H = \int\limits_{\partial M_D} d^d x \, (h^{-1}\delta h)^i_j \, \pi^j_i \,, \tag{2.28}$$

where the canonical momenta have the form

$$\pi^j_i = -\kappa \sum_{p=1}^{[(D-1)/2]} \frac{(D-2p)! \, p!}{2^{p+1}} \, \alpha_p \sum_{s=0}^{p-1} C_{s(p)} \, \left(\pi_{s(p)} \right)^j_i \,, \tag{2.29}$$

$$\left(\pi_{s(p)} \right)^j_i = \sqrt{-h} \, \delta_{[ii_1 \cdots i_{2p-1}]}^{[jj_1 \cdots j_{2p-1}]} \, \hat{R}^{i_1 i_2}_{j_1 j_2} \cdots \hat{R}^{i_{2s-1} i_{2s}}_{j_{2s-1} j_{2s}} \, K^{i_{2s+1}}_{j_{2s+1}} \cdots K^{i_{2p-1}}_{i_{2p-1}} \,, \tag{2.30}$$

and the coefficients $C_{s(p)}$ are given by

$$C_{s(p)} = \frac{4^{p-s}}{s! \, (2p-2s-1)!!} \,. \tag{2.31}$$

In the Lagrangian formalism, the variation of the action with respect to the metric defines a quasilocal (boundary) stress tensor [32], that can be therefore identified with the canonical momenta in Hamiltonian formalism. The above relations are also useful to study the generalized Israel junction conditions for branes in Lovelock gravity, as the discontinuity in the canonical momenta, and where the boundary is the brane itself [33] (for the Einstein-Gauss-Bonnet case, see [35, 34]).

JHEP10(2007)028

In Lovelock gravity with negative cosmological constant both the action and the stress tensor (or, equivalently, the canonical momenta) are in general divergent. Therefore, the regularization problem requires the addition of local counterterms, such that their inclusion does not spoil the action principle based on a Dirichlet boundary condition on the metric. For a given Lovelock-AdS theory, there is no a systematic way to generate the counterterms series, and even in the EH case it not possible to provide a closed form for $\mathcal{L}_{\rm ct}$. However, as shown in ref. [19], assuming AdS asymptotics, the divergent part of the radial canonical momenta is linked to the divergent part of the on-shell Lagrangian. The Hamilton-Jacobi relations imply that the normalizable modes of the fields expansion do not produce additional divergences and thus, the counterterms are always local. This argument opens the possibility of obtaining the Dirichlet counterterms from direct integration of the divergent parts of the Hamiltonian variation. This procedure can be performed for Chern-Simons-AdS gravity which, on the contrary to the Einstein-Hilbert case, produces a closed form for the Dirichlet counterterms (and conformal anomaly) for all odd dimensions [31]. We shall show below that the same method can be carried out (using either Hamiltonian or Lagrangian formulation) in another Lovelock theory (Born-Infeld-AdS), which can be regarded the even-dimensional counterpart of Chern-Simons-AdS, because global AdS spacetime is also a solution of maximal rank in the equations of motion.

3. Dirichlet counterterms

Let us briefly review the construction of Dirichlet counterterms for Chern-Simons-AdS gravity discussed in [31].

3.1 Chern-Simons-AdS

Chern-Simons gravity was first considered in [6] in five dimensions and in higher odd dimensions in [36, 37].

Unlike in three dimensions, higher-dimensional Chern-Simons gravity is not topological, but possesses propagating degrees of freedom [38] which number may vary from a sector to another in the phase space [39]. When the number of degrees of freedom is fewer than maximal, it is said that the sector is irregular [39]. The AdS space in pure Chern-Simons gravity is an example of such an irregular solution, and in its vicinity gravity becomes topological. However, the AdS vacuum can also be stable, as it was shown in five-dimensional Chern-Simons-AdS supergravity [40].

In Chern-Simons-AdS gravity, the Lagrangian comes from a Chern-Simons density for the group $\mathrm{SO}(2n,2)$ in $D=2n+1$ dimensions, and corresponds to the particular choice of the coefficients α_p

$$\alpha_p := \frac{\ell^{2(p-n)}}{D-2p}\binom{n}{p}\,, \qquad p \leq n\,, \tag{3.1}$$

that allows the action to be rewritten as an integration over the continuous parameter t,

$$I_{2n+1} = \kappa \int\limits_{M_{2n+1}} \int\limits_0^1 dt\, \varepsilon_{A_1\cdots A_{2n+1}} \left(\hat{R}^{A_1A_2} + \frac{t^2}{\ell^2}\, e^{A_1} e^{A_2} \right) \times$$

$$\cdots \times \left(\hat{R}^{A_{2n-1}A_{2n}} + \frac{t^2}{\ell^2}\, e^{A_{2n-1}} e^{A_{2n}} \right) e^{A_{2n+1}}\,, \tag{3.2}$$

The field equations following from the above action are

$$E_\nu^\mu = \delta^{[\mu\mu_1\cdots\mu_{2n}]}_{[\nu\nu_1\cdots\nu_{2n}]} \left(\hat{R}^{\nu_1\nu_2}_{\mu_1\mu_2} + \frac{1}{\ell^2}\, \delta^{[\nu_1\nu_2]}_{[\mu_1\mu_2]} \right) \cdots \left(\hat{R}^{\nu_{2n-1}\nu_{2n}}_{\mu_{2n-1}\mu_{2n}} + \frac{1}{\ell^2}\, \delta^{[\nu_{2n-1}\nu_{2n}]}_{[\mu_{2n-1}\mu_{2n}]} \right) = 0\,. \tag{3.3}$$

From now on, we set the AdS radius $\ell = 1$.

In order to cast the variation of the action into the form (2.28), we supplement the bulk action with the corresponding Gibbons-Hawking-Myers term

$$\beta_{2n} = -2n\sqrt{-h} \int\limits_0^1 dt \int\limits_0^1 ds\, \delta^{[i_1\ldots i_{2n-1}]}_{[j_1\ldots j_{2n-1}]}\, K^{j_1}_{i_1} \left(\frac{1}{2}\, R^{j_2j_3}_{i_2i_3}(h) - t^2 K^{j_2}_{i_2} K^{j_3}_{i_3} + s^2 \delta^{j_2}_{i_2} \delta^{j_3}_{i_3} \right) \times \cdots$$

$$\cdots \times \left(\frac{1}{2}\, R^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}}(h) - t^2 K^{j_{2n-2}}_{i_{2n-2}} K^{j_{2n-1}}_{i_{2n-1}} + s^2 \delta^{j_{2n-2}}_{i_{2n-2}} \delta^{j_{2n-1}}_{i_{2n-1}} \right). \tag{3.4}$$

Therefore, the variation of the action for the Dirichlet problem $I_{\rm Dir} = I_{2n+1} + \kappa \int_{\partial M_{2n+1}} d^{2n}x \beta_{2n}$ is given by the expression

$$\delta I^{\rm Dir}_{2n+1} = -n\kappa \int\limits_{\partial M_{2n+1}} d^{2n}x \sqrt{-h} \int\limits_0^1 dt\, \delta^{[ii_1\cdots i_{2n-1}]}_{[jj_1\cdots j_{2n-1}]}\, (h^{-1}\delta h)^j_i\, K^{j_1}_{i_1} \times$$

$$\times \left(\frac{1}{2}\, R^{j_2j_3}_{i_2i_3}(h) - t^2 K^{j_2}_{i_2} K^{j_3}_{i_3} + \delta^{j_2}_{i_2} \delta^{j_3}_{i_3} \right) \times$$

$$\cdots \times \left(\frac{1}{2}\, R^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}}(h) - t^2 K^{j_{2n-2}}_{i_{2n-2}} K^{j_{2n-1}}_{i_{2n-1}} + \delta^{j_{2n-2}}_{i_{2n-2}} \delta^{j_{2n-1}}_{i_{2n-1}} \right). \tag{3.5}$$

As we had mentioned above, this variation also defines the quasilocal stress tensor $T^i_j(h)$. In order to identify divergences and finite part of this expression, we use the Fefferman-Graham form of the metric

$$h_{ij} = \frac{1}{\rho}\, g_{ij}\,, \tag{3.6}$$

$$K^i_j = \delta^i_j - \rho\, k^i_j\,, \tag{3.7}$$

where the rescaled metric g_{ij} is given in (1.5) and $k^i_j = g^{ik}\partial_\rho g_{kj}$ are regular at the conformal boundary. Any AAdS metric can be brought into this form near $\rho = 0$. Other useful relations are

$$R^{ij}_{kl}(h) = \rho\, R^{ij}_{kl}(g)\,, \tag{3.8}$$

$$\sqrt{-h} = \frac{\sqrt{-g}}{\rho^n}\,, \tag{3.9}$$

$$(h^{-1}\delta h)^j_i = (g^{-1}\delta g)^j_i. \tag{3.10}$$

JHEP10(2007)028

It can be shown that, on the boundary, the divergent terms in (3.5) do not depend on k^i_j,

$$\delta I^{\rm Dir}_{2n+1} = -n!\kappa \int\limits_{\partial M_{2n+1}} d^{2n}x \sqrt{-g} \sum_{p=0}^{n-1} \frac{(n-p)! 2^{2n-3p-2}}{p!} \frac{1}{\rho^{n-p}} \times$$
$$\times\, \delta^{[ii_1\cdots i_{2p}]}_{[jj_1\cdots j_{2p}]} (g^{-1}\delta g)^j_i \, R^{j_1j_2}_{i_1i_2}(g) \cdots R^{j_{2p-2}j_{2p}}_{i_{2p-2}i_{2p}}(g) + \mathcal{O}(1)\,, \tag{3.11}$$

so that they can be integrated out as local functions of the boundary metric h_{ij}. These terms must be put back into the original action, with the opposite sign, playing the role of Dirichlet counterterms $\mathcal{L}_{2n}$,

$$\mathcal{L}_{2n} = n!\kappa \sqrt{-h} \sum_{p=0}^{n-1} \frac{2^{2n-3p-1}\,(n-p-1)!}{p!}\, \delta^{[i_1\cdots i_{2p}]}_{[j_1\cdots j_{2p}]} R^{j_1j_2}_{i_1i_2}(h) \cdots R^{j_{2p-2}j_{2p}}_{i_{2p-2}i_{2p}}(h)\,, \tag{3.12}$$

such that the total action

$$I^{\rm reg}_{2n+1} = I^{\rm Dir}_{2n+1} + \int\limits_{\partial M_{2n+1}} d^{2n}x\, \mathcal{L}_{2n} \tag{3.13}$$

is regularized.

The finite part in the Dirichlet variation (3.11) when $\rho \to 0$ produces a regularized stress tensor,

$$T^i_j = \frac{2}{\sqrt{-g_{(0)}}}\, g_{(0)jk} \frac{\delta I^{\rm reg}_{2n+1}}{\delta g_{(0)ki}}\,, \tag{3.14}$$

which is related to the quasilocal stress tensor $T^i_j(h)$ as

$$T^i_j = \lim_{\rho\to 0} \frac{1}{\rho^{\frac{d}{2}}}\, T^i_j(h)\,, \tag{3.15}$$

and takes the form

$$T^i_j = 2n\kappa \int\limits_0^1 dt\, \delta^{[ii_1\cdots i_{2n-1}]}_{[jj_1\cdots j_{2n-1}]}\, k^{j_1}_{i_1} \left(\frac{1}{2} R^{j_2j_3}_{i_2i_3}(g) + 2t\, k^{j_2}_{i_2} \delta^{j_3}_{i_3} \right) \times$$
$$\times \cdots \times \left(\frac{1}{2} R^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}}(g) + 2t\, k^{j_{2n-2}}_{i_{2n-2}} \delta^{j_{2n-1}}_{i_{2n-1}} \right). \tag{3.16}$$

The trace of the above stress tensor leads to a Weyl anomaly proportional to the Euler density in any $d = 2n$ dimension (type A) [31, 41]. A regularization mechanism for five-dimensional Chern-Simons-AdS gravity with Dirichlet boundary conditions, that defines a stress tensor in Riemann-Cartan spacetimes was considered in ref. [42].

In an arbitrary Lovelock gravity, the procedure of integrating out the divergent pieces as local counterterms would be more intricate because, in general, the power series in ρ would contain k^i_j, and it would be necessary to prove explicitly that there are no non-local contributions. In the Chern-Simons-AdS case, the symmetry enhancement of the theory seems to be responsible for the simple obtention of the counterterms series.

JHEP10(2007)028

3.2 Born-Infeld-AdS

Born-Infeld gravity in even dimensions ($D = 2n$) corresponds to the coefficients set

$$\alpha_p := \ell^{2(p-n)} \begin{pmatrix} n \\ p \end{pmatrix}, \qquad p \leq n-1, \tag{3.17}$$

that allows the action to be written in the form

$$\begin{aligned} I_{2n} = n\kappa \int\limits_{M_{2n}} \int\limits_0^1 du\, \varepsilon_{A_1 \cdots A_{2n}} \left(\hat{R}^{A_1 A_2} + u\, e^{A_1} e^{A_2} \right) \times \\ \cdots \times \left(\hat{R}^{A_{2n-1} A_{2n-2}} + u\, e^{A_{2n-1}} e^{A_{2n-2}} \right) e^{A_{2n-1}} e^{A_{2n}}, \end{aligned} \tag{3.18}$$

using the identity (A.3) from appendix A. The equations of motion derived from this action are

$$E_\nu^\mu = \delta^{[\mu\mu_1 \cdots \mu_{2n-2}]}_{[\nu\nu_1 \cdots \nu_{2n-2}]} \left(\hat{R}^{\nu_1\nu_2}_{\mu_1\mu_2} + \delta^{[\nu_1\nu_2]}_{[\mu_1\mu_2]} \right) \cdots \left(\hat{R}^{\nu_{2n-3}\nu_{2n-2}}_{\mu_{2n-3}\mu_{2n}-2} + \delta^{[\nu_{2n-3}\nu_{2n-2}]}_{[\mu_{2n-3}\mu_{2n-2}]} \right) = 0. \tag{3.19}$$

The generalized Gibbons-Hawking term in this case is

$$\begin{aligned} \beta_{2n-1} = -4n\,(n-1)\,\sqrt{-h} \int\limits_0^1 dt \int\limits_0^1 ds\, \delta^{[i_1 \ldots i_{2n-3}]}_{[j_1 \ldots j_{2n-3}]} K^{j_1}_{i_1} \left(\frac{1}{2} R^{j_2 j_3}_{i_2 i_3}(h) - t^2 K^{j_2}_{i_2} K^{j_3}_{i_3} + s\, \delta^{j_2}_{i_2} \delta^{j_3}_{i_3} \right) \times \\ \cdots \times \left(\frac{1}{2} R^{j_{2n-4} j_{2n-3}}_{i_{2n-4} i_{2n-3}}(h) - t^2 K^{j_{2n-4}}_{i_{2n-4}} K^{j_{2n-3}}_{i_{2n-3}} + s\, \delta^{j_{2n-4}}_{i_{2n-4}} \delta^{j_{2n-3}}_{i_{2n-3}} \right), \end{aligned} \tag{3.20}$$

and the variation of the Dirichlet action $I_{2n}^{\mathrm{Dir}} = I_{2n} + \kappa \int_{\partial M_{2n}} d^{2n-1}x\, \beta_{2n-1}$ is given by the expression

$$\begin{aligned} \delta I_{2n}^{\mathrm{Dir}} = -2n\,(n-1)\,\kappa \int\limits_{\partial M_{2n}} d^{2n-1}x\, \sqrt{-h} \int\limits_0^1 dt\, \delta^{[ii_1 \cdots i_{2n-3}]}_{[jj_1 \cdots j_{2n-3}]} \left(h^{-1}\delta h\right)^j_i K^{j_1}_{i_1} \times \\ \times \left(\frac{1}{2} R^{j_2 j_3}_{i_2 i_3}(h) - t^2 K^{j_2}_{i_2} K^{j_3}_{i_3} + \delta^{j_2}_{i_2} \delta^{j_3}_{i_3} \right) \times \cdots \\ \cdots \times \left(\frac{1}{2} R^{j_{2n-4} j_{2n-3}}_{i_{2n-4} i_{2n-3}}(h) - t^2 K^{j_{2n-4}}_{i_{2n-4}} K^{j_{2n-3}}_{i_{2n-3}} + \delta^{j_{2n-4}}_{i_{2n-4}} \delta^{j_{2n-3}}_{i_{2n-3}} \right). \end{aligned} \tag{3.21}$$

Using the Fefferman-Graham form of the metric, in the limit $\rho \to 0$, we find that the divergent terms in $\delta I_{2n}^{\mathrm{Dir}}$ do not depend on k^i_j until order $\rho^{-3/2}$,

$$\begin{aligned} \delta I_{2n}^{\mathrm{Dir}} = -n!\,\kappa \int\limits_{\partial M_{2n}} d^{2n-1}x\, \sqrt{-g} \sum_{p=0}^{n-2} \frac{2^{2n-3p-2}\,(n-p-1)!}{p!} \frac{1}{\rho^{n-p-\frac{1}{2}}} \times \\ \times \delta^{[ii_1 \cdots i_{2p}]}_{[jj_1 \cdots j_{2p}]} \left(g^{-1}\delta g\right)^j_i R^{j_1 j_2}_{i_1 i_2}(g) \cdots R^{j_{2p-1} j_{2p}}_{i_{2p-1} i_{2p}}(g) + \mathcal{O}(\rho^{-1/2}). \end{aligned} \tag{3.22}$$

Again, they can be integrated out as local functions of the boundary metric

$$\mathcal{L}_{2n-1} = n!\,\kappa \sqrt{-h} \sum_{p=0}^{n-2} \frac{2^{2n-3p-1}\,(n-p-1)!}{p!} \delta^{[i_1 \cdots i_{2p}]}_{[j_1 \cdots j_{2p}]} R^{j_1 j_2}_{i_1 i_2}(h) \cdots R^{j_{2p-1} j_{2p}}_{i_{2p-1} i_{2p}}(h), \tag{3.23}$$

JHEP10(2007)028

and should be added to the original Dirichlet action as divergent counterterms

$$I_{2n}^{\mathrm{reg}} = I_{2n}^{\mathrm{Dir}} + \int\limits_{\partial M_{2n}} d^{2n-1}x\, \mathcal{L}_{2n-1}\,. \tag{3.24}$$

What is left at the boundary, after the regularization with the Dirichlet counterterms (3.23)

$$\begin{aligned}\delta I_{2n}^{\mathrm{reg}} = \frac{2n\,(n-1)\,\kappa}{\sqrt{\rho}} \int\limits_{\partial M_{2n}} d^{2n-1}x\,\sqrt{-g} \int\limits_{0}^{1} dt\, \delta_{[jj_1\cdots j_{2n-3}]}^{[ii_1\cdots i_{2n-3}]}\,(g^{-1}\delta g)_i^j\, k_{i_1}^{j_1} \times \\ \times \left(\frac{1}{2} R_{i_2 i_3}^{j_2 j_3} + 2t\, k_{i_2}^{j_2} \delta_{i_3}^{j_3}\right) \cdots \left(\frac{1}{2} R_{i_{2n-4} i_{2n-3}}^{j_{2n-4} j_{2n-3}} + 2t\, k_{i_{2n-4}}^{j_{2n-4}} \delta_{i_{2n-3}}^{j_{2n-3}}\right)\end{aligned} \tag{3.25}$$

defines a finite stress tensor for Born-Infeld-AdS gravity, that does not provide the correct conserved quantities for static black hole solutions found in [7]. In the corresponding section below, it is shown that the difference respect a stress tensor obtained from the extrinsic regularization of the action (1.8) is at most a finite contribution.

4. Kounterterms

In the standard Dirichlet formulation of AdS gravity, the counterterms introduced to regularize the action are covariant functionals of the boundary metric, the intrinsic curvature and covariant derivatives of the intrinsic curvature. When varied, they preserve a Dirichlet boundary condition for the metric.

On the other hand, it has been recently introduced an alternative regularization procedure that consists in addition of boundary terms that contain explicit dependence on the extrinsic curvature K_{ij} (Kounterterms). This choice necessarily modifies the boundary conditions required to attain a well-posed action principle. In particular, the surface term coming from the on-shell variation of the action will contain variations of the extrinsic curvature that are usually cancelled by a generalized Gibbons-Hawking term in the Dirichlet formulation of gravity.

4.1 Chern-Simons-AdS

A boundary term that regularizes the action for Chern-Simons-AdS gravity was constructed in ref. [43], based on a well-posed action principle where the extrinsic curvature is kept fixed at the boundary. It was further clarified in [24] that this boundary condition arises naturally from the asymptotic form of the fields in Fefferman-Graham expansion. As a consequence, this condition is suitable to treat the variational problem in a large set of gravity theories that support AAdS solutions. The corresponding boundary term B_{2n} that regulates the conserved quantities and Euclidean action in Chern-Simons-AdS gravity, provides also the correct answer for Einstein-Hilbert case [22, 44], Einstein-Gauss-Bonnet gravity [23] and a generic Lovelock-AdS theory [20].

JHEP10(2007)028

We consider the Chern-Simons-AdS action in $2n+1$ dimensions,

$$\mathcal{I}_{2n+1} = I_{2n+1} + c_{2n} \int\limits_{\partial M_{2n+1}} d^{2n}x\, B_{2n}\,, \tag{4.1}$$

supplemented by a boundary term B_{2n},

$$\begin{aligned} B_{2n} \;=& -n \int\limits_0^1 dt \int\limits_0^t ds\, \varepsilon_{A_1\cdots A_{2n+1}} \theta^{A_1A_2} e^{A_3} \left(R^{A_4A_5} + t^2 (\theta^2)^{A_4A_5} + s^2 e^{A_4} e^{A_5}\right) \times \cdots \\ & \cdots \times \left(R^{A_{2n}A_{2n+1}} + t^2(\theta^2)^{A_{2n}A_{2n+1}} + s^2 e^{A_{2n}} e^{A_{2n+1}}\right)\,, \end{aligned} \tag{4.2}$$

or in a tensorial notation,

$$\begin{aligned} B_{2n} \;=& -2n\sqrt{-h} \int\limits_0^1 dt \int\limits_0^t ds\, \delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]} K^{j_1}_{i_1} \left(\frac{1}{2} R^{j_2j_3}_{i_2i_3}(h) - t^2 K^{j_2}_{i_2} K^{j_3}_{i_3} + s^2 \delta^{j_2}_{i_2} \delta^{j_3}_{i_3}\right) \times \cdots \\ & \cdots \times \left(\frac{1}{2} R^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}}(h) - t^2 K^{j_{2n-2}}_{i_{2n-2}} K^{j_{2n-1}}_{i_{2n-1}} + s^2 \delta^{j_{2n-2}}_{i_{2n-2}} \delta^{j_{2n-1}}_{i_{2n-1}}\right)\,. \end{aligned} \tag{4.3}$$

where the coupling constant takes the value $c_{2n} = \kappa$.

The explicit expression of the above boundary term can also be worked out from an extension of a Chern-Simons density (called Transgression Form) for the AdS group. This mathematical structure introduces an additional gauge connection in the same homotopy class, such that the full action is truly gauge-invariant [45].

The on-shell variation of the complete action (4.1) produces the surface term

$$\begin{aligned} \delta\mathcal{I}_{2n+1} \;=& -2n\kappa \int\limits_{\partial M_{2n+1}} \int\limits_0^1 dt\; t\, \varepsilon_{a_1\cdots a_{2n}} \left(\delta K^{a_1} e^{a_2} - K^{a_1} \delta e^{a_2}\right) \left(R^{a_3a_4} - t^2 K^{a_3} K^{a_4} + t^2 e^{a_3} e^{a_4}\right) \\ & \cdots \times \left(R^{a_{2n-1}a_{2n}} - t^2 K^{a_{2n-1}} K^{a_{2n}} + t^2 e^{a_{2n-1}} e^{a_{2n}}\right)\,, \end{aligned} \tag{4.4}$$

that, written in terms of tensors, becomes

$$\begin{aligned} \delta\mathcal{I}_{2n+1} \;=&\; 2n\kappa \int\limits_{\partial M_{2n+1}} d^{2n}x \sqrt{-h} \int\limits_0^1 dt\; t\, \delta^{[i_1\cdots i_{2n}]}_{[j_1\cdots j_{2n}]} \times \\ & \times \left(\delta K^{j_1}_{i_1} \delta^{j_2}_{i_2} + \frac{1}{2} K^{k}_{i_1} (h^{-1}\delta h)^{j_1}_{k} \delta^{j_2}_{i_2} - \frac{1}{2} K^{j_1}_{i_1} (h^{-1}\delta h)^{j_2}_{i_2}\right) \\ & \times \left(\frac{1}{2} R^{j_3j_4}_{i_3i_4}(h) - t^2 K^{j_3}_{i_3} K^{j_4}_{i_4} + t^2\, \delta^{j_3}_{i_3} \delta^{j_4}_{i_4}\right) \times \\ & \cdots \times \left(\frac{1}{2} R^{j_{2n-1}j_{2n}}_{i_{2n-1}i_{2n}}(h) - t^2 K^{j_{2n-1}}_{i_{2n-1}} K^{j_{2n}}_{i_{2n}} + t^2\, \delta^{j_{2n-1}}_{i_{2n-1}} \delta^{j_{2n}}_{i_{2n}}\right)\,. \end{aligned} \tag{4.5}$$

For an AAdS spacetime, the metric expansion (1.5) implies

$$K^i_j = \frac{1}{\ell}\, \delta^i_j - \frac{1}{\ell}\, \rho\, (g_{(1)})^i_j + \cdots\,, \tag{4.6}$$

JHEP10(2007)028

where the indices are lowered and raised by $g_{(0)ij}$. So, we will consider the condition

$$K_j^i = \frac{1}{\ell}\,\delta_j^i\,, \tag{4.7}$$

such that

$$\delta K_j^i = 0 \tag{4.8}$$

on the boundary, to cancel identically the different terms in the variation $\delta\mathcal{I}_{2n+1}$ [24].

It can be proved that the boundary term (4.3) renders the Euclidean action finite and recovers the correct black hole thermodynamics for static Chern-Simons-AdS solutions [7]. In addition, the conserved quantities can be constructed as Noether charges associated to asymptotic symmetries. However, it is clear from eq. (4.5) that this action does not lend itself to a clear definition of a boundary stress tensor, as its variation contains pieces along δK_j^i that are usually cancelled by a generalized Gibbons-Hawking term. This might make difficult the holographic interpretation of this method in the light of the AdS/CFT correspondence, where the boundary metric is kept fixed at the boundary.

Because of the delicate point mentioned above, a note of caution is in order here. The Dirichlet problem, defined as in section 2, does not really make sense for manifolds that are endowed with a conformal boundary, as it is the case of AAdS spacetimes. Indeed, the leading order of the expansion (1.5) for the boundary metric $h_{ij} = g_{ij}/\rho$ makes a Dirichlet condition inappropriate for the variational problem because of the divergence at $\rho = 0$. Thus, one should fix the conformal structure $g_{(0)ij}$ instead, and consider the addition of boundary terms to cancel the divergences at the conformal boundary. It has been argued in [19] that these boundary terms are indeed the Dirichlet counterterms, required originally by the regularization problem. This reasoning reflects an interesting connection between the boundary terms needed for a well-defined variation of the action and those that produce the action regularization. It also resembles on the regularization scheme given by eq. (1.8), where the interplay between the variational principle and the regularization problem is encoded in a single boundary term B_d.

The boundary condition (4.7) and its corresponding variation simply correspond to the regular form of the Dirichlet condition on $g_{(0)ij}$. This is a consequence of the fact that, in AAdS spacetimes, the leading order in Fefferman-Graham expansion for both the extrinsic curvature K_{ij} and the boundary metric h_{ij}/ℓ agree, what is no longer true in the flat limit $\ell \to \infty$. By selecting regular boundary conditions at $\rho = 0$, one can be certain that no additional divergences are introduced and, therefore, no extra counterterms are required on top of the series (4.3). The compatibility of this approach with keeping fixed $g_{(0)ij}$, together with the finiteness of the variation of the action, strongly suggests that the holographic reconstruction of the spacetime is already built-in in the Kounterterms series.

In what follows, we combine both the intrinsic and the extrinsic regularization mechanisms, in order to identify the Dirichlet counterterms as the difference between the Kounterterms B_{2n} and the generalized Gibbons-Hawking term β_{2n}. First, we illustrate this procedure in the five-dimensional case, where the action is

$$\mathcal{I}_5 = I_5 + \kappa \int\limits_{\partial M_5} d^4x\, B_4\,, \tag{4.9}$$

JHEP10(2007)028

with

$$B_4 = -\sqrt{-h}\delta^{[i_1 i_2 i_3]}_{[j_1 j_2 j_3]} K^{j_1}_{i_1} \left(R^{j_2 j_3}_{i_2 i_3}(h) - K^{j_2}_{i_2} K^{j_3}_{i_3} + \frac{1}{3}\, \delta^{j_2}_{i_2} \delta^{j_3}_{i_3} \right). \tag{4.10}$$

Now, let us simply insert the generalized Gibbons-Hawking term β_4 in a convenient manner,

$$\mathcal{I}_5 = I_5 + \kappa \int\limits_{\partial M_5} d^4x\, \beta_4 + \kappa \int\limits_{\partial M_5} d^4x\, \left(B_4 - \beta_4\right), \tag{4.11}$$

such that the first two terms correspond to the Dirichlet action I_5^{Dir} and will produce the finite stress tensor studied in ref. [31], plus two divergent terms

$$\delta I_5^{\mathrm{Dir}} = \frac{1}{2} \int\limits_{\partial M_5} d^4x\, \sqrt{-g}\, T^{ij}\, \delta g_{ij} - \kappa \int\limits_{\partial M_5} d^4x\, \sqrt{-g}\, (g^{-1}\delta g)^j_i \left(\frac{8}{\rho^2}\, \delta^i_j + \frac{1}{\rho}\, \delta^{[i i_1 i_2]}_{[j j_1 j_2]}\, R^{j_1 j_2}_{i_1 i_2}(g) \right). \tag{4.12}$$

Then, we compute the difference $(B_4 - \beta_4)$ as

$$(B_4 - \beta_4) = \sqrt{-h}\, \delta^{[i_1 i_2 i_3]}_{[j_1 j_2 j_3]}\, K^{j_1}_{i_1} \left(R^{j_2 j_3}_{i_2 i_3}(h) - \frac{1}{3}\, K^{j_2}_{i_2} K^{j_3}_{i_3} + \delta^{j_2}_{i_2} \delta^{j_3}_{i_3} \right), \tag{4.13}$$

and expanding the extrinsic curvature K^j_i in the radial coordinate, we realize that in the above relation, the divergent pieces do not depend on k^j_i. The different contributions can be finally seen as the local counterterms necessary to cancel the divergent terms in eq. (4.12), that is,

$$\mathcal{L}_4 = \kappa\, (B_4 - \beta_4) = 2\kappa\, \sqrt{-h} \left(8 + \delta^{[i_1 i_2]}_{[j_1 j_2]}\, R^{j_1 j_2}_{i_1 i_2}(h) \right) + \mathcal{O}(1). \tag{4.14}$$

The $\mathcal{O}(1)$ part left over at the boundary in the above difference,

$$\begin{aligned} \mathcal{L}_4^{\mathrm{fin}} &= -\kappa\, \sqrt{-g}\, \delta^{[i_1 i_2 i_3]}_{[j_1 j_2 j_3]}\, k^{j_1}_{i_1} \left(R^{j_2 j_3}_{i_2 i_3}(g) + k^{j_2}_{i_2} \delta^{j_3}_{i_3} \right) \\ &= \kappa \sqrt{-g} \left(\frac{1}{8}\, \delta^{[i_1 i_2 i_3 i_4]}_{[j_1 j_2 j_3 j_4]}\, R^{j_1 j_2}_{i_1 i_2}(g)\, R^{j_3 j_4}_{i_3 i_4}(g) + 2\delta^{[i_1 i_2]}_{[j_1 j_2]}\, k^{j_1}_{i_1} k^{j_2}_{i_2} \right), \end{aligned} \tag{4.15}$$

corresponds to the Euler-Gauss-Bonnet invariant in four dimensions plus a finite counterterm that does not contribute to the trace anomaly. (In the last line, the equation of motion (3.3), $E^\rho_\rho = 0$, was used.) This expression involves $k_{(0)ij} = g_{(1)ij}$, whose local piece has a universal form in terms of the Ricci tensor $R_{(0)ij}$ for any gravity theory with quadratic couplings in the curvature [46] (except for Chern-Simons [41]). Then, in general, this term will give rise to a quadratic combination of the curvature $R^{ij}_{(0)kl}$. This ambiguity is even present in five-dimensional Einstein-Hilbert gravity, where one can always add to the action quadratic terms in the curvature $R^{ij}_{kl}(h)$ as scheme-dependent, finite counterterms that do not modify the Weyl anomaly [15].

The same trick can be done in higher odd dimensions, such that,

$$\mathcal{I}_{2n+1} = I^{\mathrm{Dir}}_{2n+1} + \int\limits_{\partial M_{2n+1}} d^{2n}x\, \mathcal{L}_{2n}, \tag{4.16}$$

JHEP10(2007)028

where

$$\mathcal{L}_{2n} = (B_{2n} - \beta_{2n}) \tag{4.17}$$

$$\begin{aligned} &= 2n\kappa\sqrt{-h}\int_0^1 dt\int_t^1 ds\,\delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]}K^{j_1}_{i_1}\left(\frac{1}{2}R^{j_2j_3}_{i_2i_3}(h) - t^2K^{j_2}_{i_2}K^{j_3}_{i_3} + s^2\,\delta^{j_2}_{i_2}\delta^{j_3}_{i_3}\right)\times\cdots \\ &\quad\cdots\times\left(\frac{1}{2}R^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}}(h) - t^2K^{j_{2n-2}}_{i_{2n-2}}K^{j_{2n-1}}_{i_{2n-1}} + s^2\,\delta^{j_{2n-2}}_{i_{2n-2}}\delta^{j_{2n-1}}_{i_{2n-1}}\right). \end{aligned} \tag{4.18}$$

In the expansion of the extrinsic curvature $K^i_j = \delta^i_j - \rho k^i_j$ for the above expression, the divergent terms never contain k^i_j. Then, k^i_j is only present in the finite piece and terms that vanish as $\rho \to 0$. More explicitly, the expansion in $D = 7$ and $D = 9$ reads

$$\begin{aligned} \mathcal{L}_6 &= 6\kappa\sqrt{-h}\left(64 + 4\,\delta^{[i_1i_2]}_{[j_1j_2]}R^{j_1j_2}_{i_1i_2}(h) + \frac{1}{4}\,\delta^{[i_1i_2i_3i_4]}_{[j_1j_2j_3j_4]}R^{j_1j_2}_{i_1i_2}(h)\,R^{j_3j_4}_{i_3i_4}(h)\right), \\ \mathcal{L}_8 &= 24\kappa\sqrt{-h}\Big(768 + 32\,\delta^{[i_1i_2]}_{[j_1j_2]}R^{j_1j_2}_{i_1i_2}(h) + \delta^{[i_1i_2i_3i_4]}_{[j_1j_2j_3j_4]}R^{j_1j_2}_{i_1i_2}(h)R^{j_3j_4}_{i_3i_4}(h) + \\ &\qquad + \frac{1}{24}\,\delta^{[i_1i_2i_3i_4i_5i_6]}_{[j_1j_2j_3j_4j_5j_6]}R^{j_1j_2}_{i_1i_2}(h)R^{j_3j_4}_{i_3i_4}(h)R^{j_5j_6}_{i_5i_6}(h)\Big), \end{aligned}$$

up to a finite term of the type (4.15). The above examples show the agreement with the counterterms obtained from the direct integration of Dirichlet variation, eq. (3.12). Due to the lack of dependence on k^i_j, we might take directly $k^i_j = 0$ into the general expression for the counterterms (4.18), to find explicitly the terms in the Lovelock-type series

$$\mathcal{L}_{2n} = 2n\kappa\sqrt{-h}\sum_{p=0}^{n-1}\binom{n-1}{p}\frac{d_p}{2^p}\,\delta^{[i_1\ldots i_{2p}]}_{[j_1\ldots j_{2p}]}R^{j_1j_2}_{i_1i_2}(h)\ldots R^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}}(h), \tag{4.19}$$

where the coefficients are evaluated as

$$\begin{aligned} d_p &= (2n-2p)!\int_0^1 dt\int_t^1 ds\left(s^2 - t^2\right)^{n-1-p} \\ &= 4^{n-p-1}(n-p-1)!^2\,. \end{aligned} \tag{4.20}$$

In summary, the difference between the Kounterterms B_{2n} and the generalized Gibbons-Hawking term β_{2n} depends on K^i_j and might be even non-local. But, surprisingly, this procedure generates the series of local Dirichlet counterterms (3.12).

4.2 Born-Infeld-AdS

A mechanism to regularize the conserved quantities in Born-Infeld-AdS gravity in $D = 2n$ was discussed in ref. [47], where it was proposed to add the $2n$-dimensional Euler term

$$\begin{aligned} \mathcal{E}_{2n} &= \varepsilon_{A_1\cdots A_{2n}}\hat{R}^{A_1A_2}\cdots\hat{R}^{A_{2n-1}A_{2n}} \\ &= -\frac{1}{2^n}\,d^{2n}x\sqrt{-G}\,\delta^{[\mu_1\cdots\mu_{2n}]}_{[\mu_1\cdots\mu_{2n}]}\hat{R}^{\mu_1\mu_2}_{\mu_1\mu_2}\cdots\hat{R}^{\mu_{2n-1}\mu_{2n}}_{\mu_{2n-1}\mu_n} \end{aligned} \tag{4.21}$$

JHEP10(2007)028

to the bulk action (3.18). This is a topological invariant that does not modify the field equations but gives a non-trivial contribution to the Noether current. The coupling constant in front of $\mathcal{E}_{2n}$ is adjusted proceeding in the following way: let us consider the action $I_{2n} + \alpha \int_{M_{2n}} \mathcal{E}_{2n}$ (α is an arbitrary coupling constant) whose on-shell variation produces the surface term

$$\delta\left(I_{2n} + \alpha \int\limits_{M_{2n}} \mathcal{E}_{2n}\right) = n \int\limits_{\partial M_{2n}} \varepsilon_{A_1\cdots A_{2n}}\, \delta\omega^{A_1A_2} \times$$
$$\times\Big[\kappa\left(\hat{R}^{A_3A_4} + \frac{1}{\ell^2}\, e^{A_3}e^{A_4}\right)\cdots\left(\hat{R}^{A_{2n-1}A_{2n}} + \frac{1}{\ell^2}\, e^{A_{2n-1}}e^{A_{2n}}\right) +$$
$$+ (\alpha-\kappa)\ \hat{R}^{A_3A_4}\cdots\hat{R}^{A_{2n-1}A_{2n}}\Big]. \tag{4.22}$$

Therefore, demanding the spacetime to be asymptotically locally AdS, i.e.,

$$\hat{R}^{\alpha\beta}_{\mu\nu} = -\frac{1}{\ell^2}\,\delta^{[\alpha\beta]}_{[\mu\nu]} \tag{4.23}$$

at the boundary, the action is stationary on-shell only if $\alpha = \kappa$. This comes as a natural generalization of a strategy used for Einstein-Hilbert-AdS in any even dimension [47, 48].

In this way, the total action is

$$\mathcal{I}_{2n} = I_{2n} + \kappa \int\limits_{M_{2n}} \mathcal{E}_{2n}\,, \tag{4.24}$$

that takes the more compact form ($\ell = 1$)

$$\mathcal{I}_{2n} = \kappa \int\limits_{M_{2n}} \varepsilon_{A_1\cdots A_{2n}}\left(\hat{R}^{A_1A_2} + e^{A_1}e^{A_n}\right)\cdots\left(\hat{R}^{A_{2n-1}A_{2n}} + e^{A_{2n-1}}e^{A_{2n}}\right)$$
$$= -\kappa \int\limits_{M_{2n}} d^{2n}x\,\sqrt{-h}\,\delta^{[\mu_1\cdots\mu_{2n}]}_{[\nu_1\cdots\nu_{2n}]}\left(\frac{1}{2}\,\hat{R}^{\nu_1\nu_2}_{\mu_1\mu_2} + \delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2}\right)\times\cdots$$
$$\cdots\times\left(\frac{1}{2}\,\hat{R}^{\nu_{2n-1}\nu_{2n}}_{\mu_{2n-1}\mu_{2n}} + \delta^{\nu_{2n-1}}_{\mu_{2n-1}}\delta^{\nu_{2n}}_{\mu_{2n}}\right). \tag{4.25}$$

For the purpose of comparison with the Dirichlet counterterms, it is convenient to use the Euler theorem

$$\int\limits_{M_{2n}} d^{2n}x\,\mathcal{E}_{2n} = (-4\pi)^n\ n!\,\chi(M_{2n}) + \int\limits_{\partial M_{2n}} d^{2n-1}x B_{2n-1}\,, \tag{4.26}$$

to obtain the equivalence to a Kounterterms series, that is, a boundary term that depends on the extrinsic curvature K^i_j and that is given by [21]

$$B_{2n-1} = 2n\sqrt{-h}\int\limits_0^1 dt\,\delta^{[j_1\cdots j_{2n-1}]}_{[i_1\cdots i_{2n-1}]}\,K^{i_1}_{j_1}\left(\frac{1}{2}\,R^{i_2i_3}_{j_2j_3}(h) - t^2\,K^{i_2}_{j_2}K^{i_3}_{j_3}\right)\times\cdots$$
$$\cdots\times\left(\frac{1}{2}\,R^{i_{2n-2}i_{2n-1}}_{j_{2n-2}j_{2n-1}}(h) - t^2\,K^{i_{2n-2}}_{j_{2n-2}}K^{i_{2n-1}}_{j_{2n-1}}\right), \tag{4.27}$$

JHEP10(2007)028

with a coupling constant $c_{2n-1}=\kappa$.

Performing a similar procedure as in the Chern-Simons case, we add and subtract the generalized Gibbons-Hawking term into the action

$$\mathcal{I}_{2n}=I_{2n}+\kappa\int\limits_{\partial M_{2n}}d^{2n-1}x\,B_{2n-1}\,, \tag{4.28}$$

in order to identify the divergent parts,

$$\mathcal{I}_{2n}=I_{2n}^{\mathrm{Dir}}+\kappa\int\limits_{\partial M_{2n}}d^{2n-1}x\,\left(B_{2n-1}-\beta_{2n-1}\right)\,. \tag{4.29}$$

The first term in the above expression corresponds to the Dirichlet action, and the second part can be cast into the parametric integration

$$\begin{aligned}\left(B_{2n-1}-\beta_{2n-1}\right)=2n\sqrt{-h}\int\limits_0^1 dt\,\delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]}\,K^{j_1}_{i_1}\left(\frac{1}{2}\,R^{j_2j_3}_{i_2i_3}(h)-t^2K^{j_2}_{i_2}K^{j_3}_{i_3}+\delta^{j_2}_{i_2}\delta^{j_3}_{i_3}\right)\times\cdots\\ \cdots\times\left(\frac{1}{2}\,R^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}}(h)-t^2K^{j_{2n-2}}_{i_{2n-2}}K^{j_{2n-1}}_{i_{2n-1}}+\delta^{j_{2n-2}}_{i_{2n-2}}\delta^{j_{2n-1}}_{i_{2n-1}}\right)\,,\end{aligned} \tag{4.30}$$

using the identity (A.3).

Expanding the above formula using the relations (3.6)–(3.10) and the determinant of the boundary metric

$$\sqrt{-h}=\frac{\sqrt{-g}}{\rho^{n-\frac{1}{2}}}\,, \tag{4.31}$$

we notice that the divergent terms do not depend on k^i_j. As a consequence, they can be computed by setting $k^i_j=0$ and performing the integration in the parameter t, so that we have

$$\kappa\left(B_{2n-1}-\beta_{2n-1}\right)=2n\kappa\sqrt{-h}\sum_{p=0}^{n-1}\begin{pmatrix}n-1\\ p\end{pmatrix}\frac{d_p}{2^p}\,\delta^{[i_1\dots i_{2p}]}_{[j_1\dots j_{2p}]}\,R^{j_1j_2}_{i_1i_2}(h)\cdots R^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}}(h)\,, \tag{4.32}$$

where the coefficients are

$$\begin{aligned}d_p&=(2n-2p-1)!\int\limits_0^1 dt\,\left(1-t^2\right)^{n-p-1}\\ &=4^{n-p-1}(n-p-1)!^2\,.\end{aligned}$$

They can be identified, up to $\mathcal{O}(\rho^{-3/2})$, with the Dirichlet counterterms (3.23),

$$\kappa\left(B_{2n-1}-\beta_{2n-1}\right)=\mathcal{L}_{2n-1}+\frac{n\kappa}{2^{n-2}}\frac{\sqrt{-g}}{\rho^{\frac{1}{2}}}\,\delta^{[i_1\dots i_{2n-2}]}_{[j_1\dots j_{2n-2}]}\,R^{j_1j_2}_{i_1i_2}(g)\cdots R^{j_{2n-3}j_{2n-2}}_{i_{2n-3}i_{2n-2}}(g)\,. \tag{4.33}$$

In both Chern-Simons and Born-Infeld AdS gravities, if one considers flat-boundary spacetimes ($R^{ij}_{kl}(h)=0$), the Dirichlet counterterms series (eqs. (3.12) and (3.23), respectively)

JHEP10(2007)028

reduces to a single counterterm proportional to the induced volume of the boundary. Though this corresponds to a very particular case, this term is yet enough to regularize the conserved charges for horizonless extended solutions in these theories [49].

The last term of the eq. (4.33) contributes to the *finite* part of the stress tensor and, as it can be seen from the variation of the action (4.25) as[1]

$$\delta\mathcal{I}_{2n} = 2n\kappa \int\limits_{\partial M_{2n}} d^{2n-1}x\,\sqrt{-h}\,\delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]}\left(\delta K^{j_1}_{i_1} + \frac{1}{2}\,K^k_{i_1}(h^{-1}\delta h)^{j_1}_k\right)\times$$
$$\times\left(\frac{1}{2}\,R^{j_2j_3}_{i_2i_3}(h) - K^{j_2}_{i_2}K^{j_3}_{i_3} + \delta^{j_2}_{i_2}\delta^{j_3}_{i_3}\right)\times\cdots$$
$$\cdots\times\left(\frac{1}{2}\,R^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}}(h) - K^{j_{2n-2}}_{i_{2n-2}}K^{j_{2n-1}}_{i_{2n-1}} + \delta^{j_{2n-2}}_{i_{2n-2}}\delta^{j_{2n-1}}_{i_{2n-1}}\right). \qquad (4.34)$$

Indeed, counting powers of ρ, the term in eq. (4.34) along $\delta K^j_i = -\rho\,\delta k^j_i$ vanishes in the limit $\rho \to 0$, such that the stress tensor has the form

$$T^i_j(h) = 2n\kappa\,\delta^{[ki_2\cdots i_{2n-1}]}_{[jj_2\cdots j_{2n-1}]}K^i_k\left(\frac{1}{2}\,R^{j_2j_3}_{i_2i_3}(h) - K^{j_2}_{i_2}K^{j_3}_{i_3} + \delta^{j_2}_{i_2}\delta^{j_3}_{i_3}\right)\times\cdots$$
$$\cdots\times\left(\frac{1}{2}\,R^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}}(h) - K^{j_{2n-2}}_{i_{2n-2}}K^{j_{2n-1}}_{i_{2n-1}} + \delta^{j_{2n-2}}_{i_{2n-2}}\delta^{j_{2n-1}}_{i_{2n-1}}\right). \qquad (4.35)$$

The corresponding conserved quantities are constructed assuming that the boundary submanifold can be foliated in time-like ADM form

$$h_{ij}\,dx^i dx^j = -N^2_\Sigma(t)\,dt^2 + \sigma_{nm}\,(d\varphi^n + N^n_\Sigma dt)\,(d\varphi^m + N^m_\Sigma dt)\,, \qquad (4.36)$$

with the coordinates $x^i = (t, \varphi^m)$ and defined by the time-like unit normal $n_i = (-N_\Sigma, \vec{0})$. The charges are then given as the integration on Σ (the boundary of spatial section) that is parametrized by φ^m,

$$Q(\xi) = \int\limits_\Sigma d^{2n-2}\varphi\,\sqrt{\sigma}\,T^i_j(h)\,\xi^j n_i\,, \qquad (4.37)$$

where σ denotes the determinant of the metric σ_{nm} (that satisfies $\sqrt{-h} = N_\Sigma\sqrt{\sigma}$) and ξ^i is an asymptotic Killing vector. It can be verified, with the help of some of the identities extensively used above, that the conserved quantity (4.37) agrees with the charge obtained by the Noether theorem in differential forms language [47], and provides the correct mass for Born-Infeld-AdS black holes [7, 50].

Expanding the form of eq. (4.35), we notice that a finite stress tensor can be obtained multiplying $T^i_j(h)$ by a suitable factor

$$T^i_j = \lim_{\rho\to 0}\frac{1}{\rho^{\frac{d-1}{2}}}\,T^i_j(h)\,, \qquad (4.38)$$

[1]We have neglected a term along $\delta\omega^{ab}$, that can be expressed in terms of the variation of Christoffel symbol $\Gamma^i_{jk}(h) = \Gamma^i_{jk}(g)$, because it is of order $\mathcal{O}(\sqrt{\rho})$.

and can be written as

$$
\begin{aligned}
T^i_j &= n\kappa\, \delta^{[ii_1\cdots i_{2n-2}]}_{[jj_1\cdots j_{2n-2}]}\, (g^{-1}\delta g)^j_i \,\left(\frac{1}{2}\, R^{j_1 j_2}_{i_1 i_2} + 2\, k^{j_1}_{i_1}\, \delta^{j_2}_{i_2} \right) \cdots \left(\frac{1}{2}\, R^{j_{2n-3} j_{2n-2}}_{i_{2n-3} i_{2n-2}} + 2\, k^{j_{2n-3}}_{i_{2n-3}}\, \delta^{j_{2n-2}}_{i_{2n-2}} \right) \\
&= (T^i_j)_{\mathrm{Dir}} + \frac{n\kappa}{2^{n-2}}\, \delta^{[ii_1\cdots i_{2n-2}]}_{[jj_1\cdots j_{2n-2}]}\, R^{j_1 j_2}_{i_1 i_2} \cdots R^{j_{2n-3} j_{2n-2}}_{i_{2n-3} i_{2n-2}}\,,
\end{aligned}
\tag{4.39}
$$

where only the first term in $K^i_k = \delta^i_k - \rho k^i_k$ of the first line of eq. (4.35) contributes to the stress tensor. Using the components E^ρ_ρ of the equations of motion (3.19), one can prove that the trace of the above stress tensor, as expected, vanishes identically.

The first piece of the expression (4.39), $(T^i_j)_{\mathrm{Dir}}$, can be read off from the variation of the Dirichlet action (3.25). This argument shows the consistency between Dirichlet counterterms and Kounterterms also at the level of the regularized stress tensors, as they differ at most by a finite term.

5. Conclusions

In this paper, we have performed the first direct comparison between Dirichlet regularization of AdS gravity and Kounterterms prescription in two particular Lovelock theories that feature a symmetry enhancement. The remarkable agreement of the counterterms that produce the divergences cancellation in the action and stress tensor, indicates that a similar property should appear also in other Lovelock gravities with AdS asymptotics.

At this level, we simply conjecture that the Dirichlet counterterms in any Lovelock-AdS theory should be generated as the difference[2]

$$
c_d B_d - \kappa \beta_d = \mathcal{L}_d + \mathcal{O}(1)\,, \tag{5.1}
$$

though a final proof of it might be more involved than in the cases treated here.

Acknowledgments

We would like to thank M. Bañados and S. Theisen for enlightening discussions at an early stage of this work. We are also grateful to S. Detournay, D. Klemm and G. Kofinas for helpful conversations. O.M. is supported by the PUCV through the program Investigador Joven 2007. O.M. would like to thank the INFN Sezione di Milano for hospitality during the preparation of this work. The work of R.O. is supported by INFN. R.O. also thanks the organizers of the Workshop *String and M theory approaches to particle physics and cosmology* at Galileo Galilei Institute, Florence, J. Edelstein for hospitality at USC, Santiago de Compostela and G. Barnich for hospitality at ULB, Brussels.

[2] Once again, $\mathcal{O}(1)$ represents a finite term that, when d is even, does not change the trace anomaly. In turn, just because of an argument of dimensionality, when d is odd the extra term will be proportional to $1/\sqrt{\rho}$ that corresponds to a finite extra contribution to the stress tensor.

JHEP10(2007)028

A. Useful identities

The totally-antisymmetric Kronecker delta of rank m is defined as the determinant

$$\delta^{[\nu_1\cdots\nu_m]}_{[\mu_1\cdots\mu_m]} := \begin{vmatrix} \delta^{\nu_1}_{\mu_1} & \delta^{\nu_2}_{\mu_1} & \cdots & \delta^{\nu_m}_{\mu_1} \\ \delta^{\nu_1}_{\mu_2} & \delta^{\nu_2}_{\mu_2} & & \delta^{\nu_m}_{\mu_2} \\ \vdots & & \ddots & \\ \delta^{\nu_1}_{\mu_m} & \delta^{\nu_2}_{\mu_m} & \cdots & \delta^{\nu_m}_{\mu_m} \end{vmatrix} . \tag{A.1}$$

A contraction of k indices in the above Kronecker delta produces a delta of order $m-k$,

$$\delta^{[\nu_1\cdots\nu_k\cdots\nu_m]}_{[\mu_1\cdots\mu_k\cdots\mu_m]}\,\delta^{\mu_1}_{\nu_1}\cdots\delta^{\mu_k}_{\nu_k} = \frac{(N-m+k)!}{(N-m)!}\,\delta^{[\nu_{k+1}\cdots\nu_m]}_{[\mu_{k+1}\cdots\mu_m]}\,, \qquad (1\leq k\leq m\leq N)\,, \tag{A.2}$$

where N is the range of indices.

A useful identity that has been employed in the paper involves the binomial expansion given in an integral form,

$$(a+b)^p = a^p + p\,b\int_0^1 du\;(a+ub)^{p-1}\;, \qquad p\geq 1\,. \tag{A.3}$$

Other two integral representations of a binomial often used in the text are

$$\int_0^1 dt\;\left[a+(2p+1)\,t^2b\right]\left(a+t^2b^2\right)^{p-1} = (a+b)^p\;, \qquad p\geq 1\,, \tag{A.4}$$

$$\int_0^1 dt\,2t\left[a+(p+1)\,t^2b\right]\left(a+t^2b\right)^{p-1} = (a+b)^p\,, \qquad p\geq 1\,. \tag{A.5}$$

References

[1] C. Lanczos, *A remarkable property of the riemann-christoffel tensor in four dimensions*, *Annals Math.* **39** (1938) 842;
D. Lovelock, *The Einstein tensor and its generalizations*, *J. Math. Phys.* **12** (1971) 498; *The four-dimensionality of space and the Einstein tensor*, *J. Math. Phys.* **13** (1972) 874.

[2] B. Zumino, *Gravity theories in more than four-dimensions*, *Phys. Rept.* **137** (1986) 109.

[3] D.G. Boulware and S. Deser, *String generated gravity models*, *Phys. Rev. Lett.* **55** (1985) 2656;
J.T. Wheeler, *Symmetric solutions to the maximally Gauss-Bonnet extended Einstein equations*, *Nucl. Phys.* **B 273** (1986) 732; *Symmetric solutions to the Gauss-Bonnet extended Einstein equations*, *Nucl. Phys.* **B 268** (1986) 737.

[4] R.C. Myers and J.Z. Simon, *Black hole thermodynamics in Lovelock gravity*, *Phys. Rev.* **D 38** (1988) 2434.

[5] J. Crisostomo, R. Troncoso and J. Zanelli, *Black hole scan*, *Phys. Rev.* **D 62** (2000) 084013 [hep-th/0003271].

JHEP10(2007)028

[6] A.H. Chamseddine, *Topological gauge theory of gravity in five-dimensions and all odd dimensions*, *Phys. Lett.* **B 233** (1989) 291.

[7] M. Bañados, C. Teitelboim and J. Zanelli, *Dimensionally continued black holes*, *Phys. Rev.* **D 49** (1994) 975 [gr-qc/9307033].

[8] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200];
S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys.* **B 636** (2002) 99 [hep-th/0204051];
E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[9] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [hep-th/9806087].

[10] S. de Haro, S.N. Solodukhin and K. Skenderis, *Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595 [hep-th/0002230].

[11] K. Skenderis, *Lecture notes on holographic renormalization*, *Class. and Quant. Grav.* **19** (2002) 5849 [hep-th/0209067].

[12] For counterterms in non AAdS spactimes, see R.-G. Cai and N. Ohta, *Surface counterterms and boundary stress-energy tensors for asymptotically non-Anti-de Sitter spaces*, *Phys. Rev.* **D 62** (2000) 024006 [hep-th/9912013].

[13] C. Fefferman and C.R. Graham, *Conformal invariants*, in *The mathematical heritage of Elie Cartan*, Lyon (1984), Astérisque **95** (1985).

[14] G.W. Gibbons and S.W. Hawking, *Action integrals and partition functions in quantum gravity*, *Phys. Rev.* **D 15** (1977) 2752.

[15] V. Balasubramanian and P. Kraus, *A stress tensor for Anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413 [hep-th/9902121].

[16] R. Emparan, C.V. Johnson and R.C. Myers, *Surface terms as counterterms in the AdS/CFT correspondence*, *Phys. Rev.* **D 60** (1999) 104001 [hep-th/9903238];
R.B. Mann, *Misner string entropy*, *Phys. Rev.* **D 60** (1999) 1040047 [hep-th/9903229].

[17] S. Nojiri, S.D. Odintsov and S. Ogushi, *Cosmological and black hole brane world universes in higher derivative gravity*, *Phys. Rev.* **D 65** (2002) 023521 [hep-th/0108172].

[18] M. Cvetič, S. Nojiri and S.D. Odintsov, *Black hole thermodynamics and negative entropy in desitter and Anti-de Sitter Einstein-Gauss-Bonnet gravity*, *Nucl. Phys.* **B 628** (2002) 295 [hep-th/0112045].

[19] I. Papadimitriou and K. Skenderis, *AdS/CFT correspondence and geometry*, hep-th/0404176; *Thermodynamics of asymptotically locally AdS spacetimes*, *JHEP* **08** (2005) 004 [hep-th/0505190].

[20] G. Kofinas and R. Olea, *Universal regularization prescription for Lovelock AdS gravity*, arXiv:0708.0782.

[21] R. Olea, *Mass, angular momentum and thermodynamics in four-dimensional Kerr-AdS black holes*, *JHEP* **06** (2005) 023 [hep-th/0504233].

JHEP10(2007)028

[22] R. Olea, *Regularization of odd-dimensional AdS gravity: kounterterms*, *JHEP* **04** (2007) 073 [hep-th/0610230].

[23] G. Kofinas and R. Olea, *Vacuum energy in Einstein-Gauss-Bonnet AdS gravity*, *Phys. Rev.* **D 74** (2006) 084035 [hep-th/0606253].

[24] O. Mišković and R. Olea, *On boundary conditions in three-dimensional AdS gravity*, *Phys. Lett.* **B 640** (2006) 101 [hep-th/0603092].

[25] R.C. Myers, *Higher derivative gravity, surface terms and string theory*, *Phys. Rev.* **D 36** (1987) 392.

[26] T. Eguchi, P.B. Gilkey and A.J. Hanson, *Gravitation, gauge theories and differential geometry*, *Phys. Rept.* **66** (1980) 213.

[27] M. Nakahara, *Geometry, topology and physics*, IOP, Bristol (1991).

[28] Y. Choquet-Bruhat and C. Dewitt-Morette, *Analysis, manifolds and physics*, North Holland (2001).

[29] F. Mueller-Hoissen, *Gravity actions, boundary terms and second order field equations*, *Nucl. Phys.* **B 337** (1990) 709.

[30] H. Stephani, *Some perfect fluid solutions of Einstein's field equations without symmetries*, *Class. and Quant. Grav.* **4** (1987) 125.

[31] M. Bañados, R. Olea and S. Theisen, *Counterterms and dual holographic anomalies in CS gravity*, *JHEP* **10** (2005) 067 [hep-th/0509179].

[32] J.D. Brown and J. York, James W., *Quasilocal energy and conserved charges derived from the gravitational action*, *Phys. Rev.* **D 47** (1993) 1407.

[33] E. Gravanis and S. Willison, *Intersecting hypersurfaces, topological densities and Lovelock gravity*, *J. Geom. Phys.* **57** (2007) 1861 [gr-qc/0401062].

[34] E. Gravanis and S. Willison, *Israel conditions for the Gauss-Bonnet theory and the Friedmann equation on the brane universe*, *Phys. Lett.* **B 562** (2003) 118 [hep-th/0209076].

[35] S.C. Davis, *Generalised israel junction conditions for a Gauss-Bonnet brane world*, *Phys. Rev.* **D 67** (2003) 024030 [hep-th/0208205].

[36] A.H. Chamseddine, *Topological gravity and supergravity in various dimensions*, *Nucl. Phys.* **B 346** (1990) 213.

[37] R. Troncoso and J. Zanelli, *New gauge supergravity in seven and eleven dimensions*, *Phys. Rev.* **D 58** (1998) 101703 [hep-th/9710180]; *Gauge supergravities for all odd dimensions*, *Int. J. Theor. Phys.* **38** (1999) 1181 [hep-th/9807029].

[38] M. Bañados, L.J. Garay and M. Henneaux, *The dynamical structure of higher dimensional Chern-Simons theory*, *Nucl. Phys.* **B 476** (1996) 611 [hep-th/9605159].

[39] O. Mišković and J. Zanelli, *Dynamical structure of irregular constrained systems*, *J. Math. Phys.* **44** (2003) 3876 [hep-th/0302033];
O. Mišković, R. Troncoso and J. Zanelli, *Canonical sectors of five-dimensional Chern-Simons theories*, *Phys. Lett.* **B 615** (2005) 277 [hep-th/0504055].

[40] O. Mišković, R. Troncoso and J. Zanelli, *Dynamics and BPS states of AdS_5 supergravity with a Gauss-Bonnet term*, *Phys. Lett.* **B 637** (2006) 317 [hep-th/0603183].

JHEP10(2007)028

[41] M. Bañados, A. Schwimmer and S. Theisen, *Chern-Simons gravity and holographic anomalies*, *JHEP* **05** (2004) 039 [hep-th/0404245].

[42] M. Bañados, O. Mišković and S. Theisen, *Holographic currents in first order gravity and finite Fefferman-Graham expansions*, *JHEP* **06** (2006) 025 [hep-th/0604148].

[43] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Finite action principle for Chern-Simons AdS gravity*, *JHEP* **06** (2004) 036 [hep-th/0405267].

[44] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Vacuum energy in odd-dimensional AdS gravity*, hep-th/0412046.

[45] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Transgression forms and extensions of Chern-Simons gauge theories*, *JHEP* **02** (2006) 067 [hep-th/0601081].

[46] C. Imbimbo, A. Schwimmer, S. Theisen and S. Yankielowicz, *Diffeomorphisms and holographic anomalies*, *Class. and Quant. Grav.* **17** (2000) 1129 [hep-th/9910267].

[47] R. Aros, M. Contreras, R. Olea, R. Troncoso and J. Zanelli, *Conserved charges for even dimensional asymptotically AdS gravity theories*, *Phys. Rev.* **D 62** (2000) 044002 [hep-th/9912045].

[48] R. Aros, M. Contreras, R. Olea, R. Troncoso and J. Zanelli, *Conserved charges for gravity with locally AdS asymptotics*, *Phys. Rev. Lett.* **84** (2000) 1647 [gr-qc/9909015].

[49] M.H. Dehghani, N. Bostani and A. Sheikhi, *Counterterm method in Lovelock theory and horizonless solutions in dimensionally continued gravity*, *Phys. Rev.* **D 73** (2006) 104013 [hep-th/0603058].

[50] R.G. Cai and K.-S. Soh, *Topological black holes in the dimensionally continued gravity*, *Phys. Rev.* **D 59** (1999) 044013 [gr-qc/9808067].

JHEP10(2007)028